Principle of Mathematical Induction

A given statement $S(n)$ is true for each positive integer n if two below conditions hold

Condition I: $S(1)$ is true i.e. $S(n)$ is true for $n = 1$ and

Condition II: $S(k+1)$ is true whenever $S(k)$ is true for any positive integer k,

Then $S(n)$ is true for all positive integers.

Use mathematical induction to prove the following formulae for every positive integer n

Question # 1

$$1+5+9+...+(4n-3) = n(2n-1)$$

Solution Suppose $S(n): 1+5+9+..............+(4n-3) = n(2n-1)$

Put $n = 1$

$\quad S(1): 1 = 1(2(1)-1) \implies 1 = 1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$\quad S(k): 1+5+9+..............+(4k-3) = k(2k-1)$ ………… (i)

The statement for $n = k+1$ becomes

$\quad S(k+1): 1+5+9+..............+(4(k+1)-3) = (k+1)(2(k+1)-1)$

$\quad \implies 1+5+9+..............+(4k+1) = (k+1)(2k+2-1)$

$$= (k+1)(2k+1)$$

$$= 2k^2 + 2k + k + 1$$

$$= 2k^2 + 3k + 1$$

Adding $4k+1$ on both sides of equation (i)

$$1+5+9+..............+(4k-3)+(4k+1) = k(2k-1)+4k+1$$

$$\implies 1+5+9+..............+(4k+1) = 2k^2 - k + 4k + 1$$

$$= 2k^2 + 3k + 1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 2

$$1+3+5+...+(2n-1) = n^2$$

Solution Suppose

$\quad S(n): 1+3+5+..............+(2n-1) = n^2$

Put $n = 1$

$\quad S(1): 1 = (1)^2 \implies 1 = 1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$\quad S(k): 1+3+5+..............+(2k-1) = k^2$ ………… (i)

The statement for $n = k+1$ becomes

$$S(k+1): \ 1+3+5+\ldots\ldots\ldots+(2(k+1)-1)=(k+1)^2$$

$$\Rightarrow \ 1+3+5+\ldots\ldots\ldots+(2k+1)=(k+1)^2$$

Adding $2k+1$ on both sides of equation (i)

$$1+3+5+\ldots\ldots\ldots+(2k-1)+(2k+1)=k^2+2k+1$$

$$\Rightarrow \ 1+3+5+\ldots\ldots\ldots+(2k+1)=(k+1)^2$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 3

$$1+4+7+\ldots+(3n-2)=\frac{n(3n-1)}{2}$$

Solution   Suppose

$$S(n): \ 1+4+7+\ldots\ldots\ldots+(3n-2)=\frac{n(3n-1)}{2}$$

Put $n=1$

$$S(1): \ 1=\frac{1(3(1)-1)}{2} \quad \Rightarrow \ 1=\frac{2}{2} \quad \Rightarrow 1=1$$

Thus condition I is satisfied
Now suppose that $S(n)$ is true for $n=k$

$$S(k): \ 1+4+7+\ldots\ldots\ldots+(3k-2)=\frac{k(3k-1)}{2} \quad \ldots\ldots\ldots\ldots \text{ (i)}$$

The statement for $n=k+1$ becomes

$$S(k+1): \ 1+4+7+\ldots\ldots\ldots+(3(k+1)-2)=\frac{(k+1)(3(k+1)-1)}{2}$$

$$\Rightarrow \ 1+4+7+\ldots\ldots\ldots+(3k+1)=\frac{(k+1)(3k+3-1)}{2}$$

$$=\frac{(k+1)(3k+2)}{2}$$

Adding $3k+1$ on both sides of equation (i)

$$1+4+7+\ldots\ldots\ldots+(3k-2)+(3k+1)=\frac{k(3k-1)}{2}+3k+1$$

$$\Rightarrow \ 1+4+7+\ldots\ldots\ldots+(3k+1)=\frac{k(3k-1)+2(3k+1)}{2}$$

$$=\frac{3k^2-k+6k+2}{2}$$

$$=\frac{3k^2+5k+2}{2}$$

$$=\frac{3k^2+3k+2k+2}{2}$$

$$=\frac{3k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(3k+2)}{2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

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Question # 4

$$1+2+4+...+2^n = 2^n -1$$

Solution   Suppose

$$S(n):\ 1+2+4+.............+2^{n-1} = 2^n -1$$

Put $n = 1$

$$S(1):\ 1 = 2^1 -1 \ \Rightarrow \ 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k):\ 1+2+4+.............+2^{k-1} = 2^k -1 \ .... \ (i)$$

The statement for $n = k+1$ becomes

$$S(k+1):\ 1+2+4+.............+2^{k+1-1} = 2^{k+1} -1$$

$$\Rightarrow \ 1+2+4+.............+2^k = 2^{k+1} -1$$

adding 2^k on both sides of equation (i)

$$1+2+4+.............+2^{k-1}+2^k = 2^k -1+2^k$$

$$\Rightarrow \ 1+2+4+.............+2^k = 2(2^k)-1 \qquad\qquad \because \ 2^k + 2^k = 2(2^k)$$

$$= 2^{k+1} -1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

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Question # 5

$$1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^{n-1}} = 2\left[1-\frac{1}{2^n}\right]$$

Solution   Suppose

$$S(n):\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{2^{n-1}} = 2\left(1-\frac{1}{2^n}\right)$$

Put $n = 1$

$$S(1):\ 1 = 2\left(1-\frac{1}{2^1}\right) \ \Rightarrow \ 1 = 2\left(\frac{1}{2}\right) \ \Rightarrow \ 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k):\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{2^{k-1}} = 2\left(1-\frac{1}{2^k}\right) \ .... \ (i)$$

The statement for $n = k+1$ becomes

$$S(k+1):\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{2^{k+1-1}} = 2\left(1-\frac{1}{2^{k+1}}\right)$$

$$\Rightarrow \ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{2^k} = 2-\frac{2}{2^{k+1}}$$

$$= 2 - \frac{2}{2^k \cdot 2}$$

$$= 2 - \frac{1}{2^k}$$

Adding $\dfrac{1}{2^k}$ on both sides of equation (i)

$$1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots\ldots + \frac{1}{2^{k-1}} + \frac{1}{2^{k+1-1}} = 2\left(1 - \frac{1}{2^k}\right) + \frac{1}{2^k}$$

$$\Rightarrow \quad 1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots\ldots + \frac{1}{2^{k+1-1}} = 2 - \frac{2}{2^k} + \frac{1}{2^k}$$

$$= 2 - \frac{1}{2^k}(2 - 1)$$

$$= 2 - \frac{1}{2^k}(1) = 2 - \frac{1}{2^k}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 6　　　　　　　*Do yourself as Question # 1*

Question # 7

$$2 + 6 + 18 + \ldots + 2 \times 3^{n-1} = 3^n - 1$$

Solution　　Suppose

$$S(n): \ 2 + 6 + 18 + \ldots\ldots\ldots + 2 \times 3^{n-1} = 3^n - 1$$

Put $n = 1$

$$S(1): \ 2 = 3^1 - 1 \ \Rightarrow \ 2 = 2$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \ 2 + 6 + 18 + \ldots\ldots\ldots + 2 \times 3^{k-1} = 3^k - 1 \ \ldots\ldots\ldots\ldots \ \text{(i)}$$

The statement for $n = k + 1$ becomes

$$S(k+1): \ 2 + 6 + 18 + \ldots\ldots\ldots + 2 \times 3^{k+1-1} = 3^{k+1} - 1$$

Adding 2×3^k on both sides of equation (i)

$$2 + 6 + 18 + \ldots\ldots\ldots + 2 \times 3^{k-1} + 2 \times 3^k = 3^k - 1 + 2 \times 3^k$$

$$\Rightarrow \quad 2 + 6 + 18 + \ldots\ldots\ldots + 2 \times 3^{k+1-1} = 3(3^k) - 1 \qquad\qquad \because \ 3^k + 2 \times 3^k = 3(3^k)$$

$$= 3^{k+1} - 1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all　positive integer n.

Question # 8

$$1 \times 3 + 2 \times 5 + 3 \times 7 + \ldots + n \times (2n+1) = \frac{n(n+1)(4n+5)}{6}$$

Solution　　Suppose

$$S(n): 1 \times 3 + 2 \times 5 + 3 \times 7 + \ldots\ldots\ldots + n \times (2n+1) = \frac{n(n+1)(4n+5)}{6}$$

Put $n = 1$

$$S(1): \ 1\times 3 = \frac{1(1+1)(4(1)+5)}{6} \quad \Rightarrow \quad 3 = \frac{(2)(9)}{6} \quad \Rightarrow \quad 3 = 3$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \ 1\times 3 + 2\times 5 + 3\times 7 +\ldots\ldots\ldots+k\times(2k+1) = \frac{k(k+1)(4k+5)}{6} \quad \ldots\ldots \text{ (i)}$$

The statement for $n = k+1$ becomes

$$S(k+1): 1\times 3 + 2\times 5 + 3\times 7 +\ldots\ldots\ldots+(k+1)\times(2(k+1)+1) = \frac{(k+1)(k+1+1)(4(k+1)+5)}{6}$$

$$\Rightarrow \ 1\times 3 + 2\times 5 + 3\times 7 +\ldots\ldots\ldots+(k+1)\times(2k+3) = \frac{(k+1)(k+2)(4k+9)}{6}$$

Adding $(k+1)\times(2k+3)$ on both sides of equation (i)

$$1\times 3 + 2\times 5 + 3\times 7 +\ldots\ldots\ldots+k\times(2k+1)+(k+1)\times(2k+3) = \frac{k(k+1)(4k+5)}{6}+(k+1)\times(2k+3)$$

$$\Rightarrow \ 1\times 3 + 2\times 5 + 3\times 7 +\ldots\ldots\ldots+(k+1)\times(2k+3) = (k+1)\left(\frac{k(4k+5)}{6}+(2k+3)\right)$$

$$= (k+1)\left(\frac{k(4k+5)+6(2k+3)}{6}\right)$$

$$= (k+1)\left(\frac{4k^2+5k+12k+18}{6}\right)$$

$$= (k+1)\left(\frac{4k^2+17k+18}{6}\right)$$

$$= (k+1)\left(\frac{4k^2+17k+18}{6}\right)$$

$$= (k+1)\left(\frac{4k^2+8k+9k+18}{6}\right)$$

$$= (k+1)\left(\frac{4k(k+2)+9(k+2)}{6}\right)$$

$$= (k+1)\left(\frac{(k+2)(4k+9)}{6}\right)$$

$$= \frac{(k+1)(k+2)(4k+9)}{6}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 9

$$1\times 2 + 2\times 3 + 3\times 4 + \ldots + n\times(n+1) = \frac{n(n+1)(n+2)}{3}$$

Solution   *Do yourself as Question # 8*

Question # 10

$$1\times2+3\times4+5\times6+...+(2n-1)\times2n=\frac{n(n+1)(4n-1)}{3}$$

Solution   *Do yourself as Question # 8*

Question # 11

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+...+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

Solution   Suppose

$$S(n):\ \frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+.............+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

Put *n* = 1

$$S(1):\ \frac{1}{1\times2}=1-\frac{1}{1+1}\ \Rightarrow\ \frac{1}{2}=1-\frac{1}{2}\ \Rightarrow\ \frac{1}{2}=\frac{1}{2}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ \frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+.............+\frac{1}{k(k+1)}=1-\frac{1}{k+1}\ \ ....\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{(k+1)(k+1+1)}=1-\frac{1}{k+1+1}$$

$$\Rightarrow\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+2}$$

Adding $\dfrac{1}{(k+1)(k+2)}$ on both sides of equation (i)

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+.............+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$\Rightarrow\ \frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+.............+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1}\left(1-\frac{1}{(k+2)}\right)$$

$$=1-\frac{1}{k+1}\left(\frac{k+2-1}{k+2}\right)$$

$$=1-\frac{1}{k+1}\left(\frac{k+1}{k+2}\right)$$

$$=1-\frac{1}{k+2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer *n*.

Question # 12

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+............+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Solution   Suppose

$$S(n): \frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \ldots\ldots\ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Put $n = 1$

$$S(1): \frac{1}{1\times3} = \frac{1}{2(1)+1} \quad \Rightarrow \quad \frac{1}{3} = \frac{1}{3}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \ldots\ldots\ldots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1} \quad \ldots\ldots\ldots \text{ (i)}$$

The statement for $n = k+1$ becomes

$$S(k+1): \frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \ldots\ldots\ldots + \frac{1}{\big(2(k+1)-1\big)\big(2(k+1)+1\big)} = \frac{k+1}{2(k+1)+1}$$

$$\Rightarrow \quad \frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \ldots\ldots\ldots + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$$

Adding $\dfrac{1}{(2k+1)(2k+3)}$ on both sides of equation (i)

$$\frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \ldots\ldots\ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$\Rightarrow \quad \frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \ldots\ldots\ldots + \frac{1}{(2k+1)(2k+3)} = \frac{1}{2k+1}\left(k + \frac{1}{(2k+3)}\right)$$

$$= \frac{1}{2k+1}\left(\frac{k(2k+3)+1}{2k+3}\right)$$

$$= \frac{1}{2k+1}\left(\frac{2k^2+3k+1}{2k+3}\right)$$

$$= \frac{1}{2k+1}\left(\frac{2k^2+2k+k+1}{2k+3}\right)$$

$$= \frac{1}{2k+1}\left(\frac{2k(k+1)+1(k+1)}{2k+3}\right)$$

$$= \frac{1}{2k+1}\left(\frac{(2k+1)(k+1)}{2k+3}\right)$$

$$= \left(\frac{k+1}{2k+3}\right)$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 13

$$\frac{1}{2\times5} + \frac{1}{5\times8} + \frac{1}{8\times11} + \ldots\ldots\ldots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{2(3n+2)}$$

Solution   *Do yourself as Question # 12*

Question # 14

$$r + r^2 + r^3 + \ldots\ldots\ldots + r^n = \frac{r(1-r^n)}{1-r} \qquad (r \neq 1)$$

Solution   Suppose $S(n)$: $r + r^2 + r^3 + \ldots\ldots\ldots + r^n = \frac{r(1-r^n)}{1-r}$

Put $n = 1$

$$S(1): \; r = \frac{r\left(1-r^1\right)}{1-r} \;\Rightarrow\; r = r$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \; r + r^2 + r^3 + \ldots\ldots\ldots + r^k = \frac{r(1-r^k)}{1-r} \quad \ldots\ldots\ldots\ldots \text{ (i)}$$

The statement for $n = k+1$ becomes

$$S(k+1): \; r + r^2 + r^3 + \ldots\ldots\ldots + r^{k+1} = \frac{r(1-r^{k+1})}{1-r}$$

Adding r^{k+1} on both sides of equation (i)

$$r + r^2 + r^3 + \ldots\ldots\ldots + r^k + r^{k+1} = \frac{r(1-r^k)}{1-r} + r^{k+1}$$

$$\Rightarrow \; r + r^2 + r^3 + \ldots\ldots\ldots + r^{k+1} = \frac{r(1-r^k) + r^{k+1}(1-r)}{1-r}$$

$$= \frac{r - r^{k+1} + r^{k+1} - r^{k+2}}{1-r}$$

$$= \frac{r - r^{k+2}}{1-r}$$

$$= \frac{r(1-r^{k+1})}{1-r}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 15

$$a + (a+d) + (a+2d) + \ldots\ldots\ldots + [a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$$

Solution   Suppose

$$S(n): \; a + (a+d) + (a+2d) + \ldots\ldots\ldots + [a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$$

Put $n = 1$

$$S(1): \; a = \frac{1}{2}[2a + (1-1)d] \;\Rightarrow\; a = \frac{1}{2}[2a + (0)d] \;\Rightarrow\; a = \frac{1}{2}[2a] = a$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \ a+(a+d)+(a+2d)+\ldots\ldots\ldots+\left[a+(k-1)d\right]=\frac{k}{2}\left[2a+(k-1)d\right] \ \ldots\ldots\text{ (i)}$$

The statement for $n=k+1$ becomes

$$S(k+1): \ a+(a+d)+(a+2d)+\ldots\ldots\ldots+\left[a+(k+1-1)d\right]=\frac{k+1}{2}\left[2a+(k+1-1)d\right]$$

$$\Rightarrow \ a+(a+d)+(a+2d)+\ldots\ldots\ldots+\left[a+kd\right]=\frac{k+1}{2}\left[2a+kd\right]$$

Adding $a+kd$ on both sides of equation (i)

$$a+(a+d)+(a+2d)+\ldots\ldots\ldots+\left[a+(k-1)d\right]+\left[a+kd\right]=\frac{k}{2}\left[2a+(k-1)d\right]+\left[a+kd\right]$$

$$\Rightarrow \ a+(a+d)+(a+2d)+\ldots\ldots\ldots+\left[a+kd\right]=\frac{k}{2}\left[2a+kd-d\right]+\left[a+kd\right]$$

$$=\frac{k\left[2a+kd-d\right]+2\left[a+kd\right]}{2}$$

$$=\frac{2ak+k^2d-kd+2a+2kd}{2}$$

$$=\frac{2ak+k^2d+kd+2a}{2}$$

$$=\frac{2ak+2a+k^2d+kd}{2}$$

$$=\frac{2a(k+1)+kd(k+1)}{2}$$

$$=\frac{(k+1)(2a+kd)}{2}$$

$$=\frac{k+1}{2}\left[2a+kd\right]$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 16

$$1\cdot\lfloor 1+2\cdot\lfloor 2+3\cdot\lfloor 3+\ldots\ldots\ldots+n\cdot\lfloor n=\lfloor n+1-1$$

Solution

Suppose $S(n): \ 1\cdot\lfloor 1+2\cdot\lfloor 2+3\cdot\lfloor 3+\ldots\ldots\ldots+n\cdot\lfloor n=\lfloor n+1-1$

Put $n=1$

$S(1): \ 1\cdot\lfloor 1=\lfloor 1+1-1 \ \Rightarrow \ 1=\lfloor 2-1 \ \Rightarrow \ 1=2-1 \ \Rightarrow \ 1=1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$S(k): \ 1\cdot\lfloor 1+2\cdot\lfloor 2+3\cdot\lfloor 3+\ldots\ldots\ldots+k\cdot\lfloor k=\lfloor k+1-1 \ \ldots\ldots\ldots\text{ (i)}$

The statement for $n=k+1$ becomes

$S(k+1): \ 1\cdot\lfloor 1+2\cdot\lfloor 2+3\cdot\lfloor 3+\ldots\ldots\ldots+(k+1)\cdot\lfloor k+1=\lfloor k+1+1-1$

$\Rightarrow \ 1\cdot\lfloor 1+2\cdot\lfloor 2+3\cdot\lfloor 3+\ldots\ldots\ldots+(k+1)\cdot\lfloor k+1=\lfloor k+2-1$

Adding $(k+1)\cdot\lfloor k+1$ on both sides of equation (i)

$1\cdot\lfloor 1+2\cdot\lfloor 2+3\cdot\lfloor 3+\ldots\ldots\ldots+k\cdot\lfloor k+(k+1)\cdot\lfloor k+1=\lfloor k+1-1+(k+1)\cdot\lfloor k+1$

$$\Rightarrow \quad 1\cdot\lfloor 1 + 2\cdot\lfloor 2 + 3\cdot\lfloor 3 + \ldots\ldots\ldots\ldots + (k+1)\cdot\lfloor k+1 = \lfloor k+1 + \lfloor k+1(k+1) - 1$$
$$= \lfloor k+1\,(1+k+1) - 1$$
$$= \lfloor k+1\,(k+2) - 1$$
$$= (k+2)\lfloor k+1 - 1$$
$$= \lfloor k+2 - 1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 17

$$a_n = a_1 + (n-1)d \qquad\qquad \text{When, } a_1, a_1+d, a_1+2d,\ldots \text{ from an A.P.}$$

Solution

Suppose $\qquad S(n): \ a_n = a_1 + (n-1)d$

Put $n=1$

$S(1): \ a_1 = a_1 + (1-1)d \ \Rightarrow\ a_1 = a_1 + 0d = a_1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$\qquad S(k): \ a_k = a_1 + (k-1)d$ …………. (i)

The statement for $n = k+1$ becomes

$\qquad S(k+1): \ a_{k+1} = a_1 + (k+1-1)d$
$$\qquad\qquad = a_1 + (k)d$$

Adding d on both sides of equation (i)

$$a_k + d = a_1 + (k-1)d + d \qquad\qquad \because\ a_2 = a_1 + d$$
$$\Rightarrow\ a_{k+1} = a_1 + (k-1+1)d \qquad\qquad\quad a_3 = a_2 + d$$
$$\Rightarrow\ a_{k+1} = a_1 + (k)d \qquad\qquad\qquad\quad \therefore\ a_{k+1} = a_k + d$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 18

$$a_n = a_1 r^{n-1} \qquad\qquad \text{When, } a_1, a_1 r, a_1 r^2,\ldots \text{ from an G.P.}$$

Solution Suppose $\qquad\qquad S(n): \ a_n = a_1 r^{n-1}$

Put $n=1$

$S(1): \ a_1 = a_1 r^{1-1} \ \Rightarrow\ a_1 = a_1 r^0 = a_1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$\qquad S(k): \ a_k = a_1 r^{k-1}$ …………. (i)

The statement for $n = k+1$ becomes

$\qquad S(k+1): \ a_{k+1} = a_1 r^{k+1-1}$
$$\qquad\qquad = a_1 r^k$$

Multiplying r on both sides of equation (i)

$$a_k \cdot r = a_1 r^{k-1} \cdot r^1 \qquad\qquad \because\ a_2 = a_1 r$$
$$\Rightarrow\ a_{k+1} = a_1 r^{k-1+1} \qquad\qquad\quad a_3 = a_2 r$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\quad \therefore\ a_{k+1} = a_k r$$

$$\Rightarrow \quad a_{k+1} = a_1 r^k$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer *n*.

Question # 19

$$1^2 + 3^2 + 5^2 + \ldots\ldots\ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

Solution

Suppose $\qquad S(n): 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$

Put *n* = 1

$$S(1): 1^2 = \frac{1(4(1)^2-1)}{3} \quad \Rightarrow \quad 1 = \frac{1(4-1)}{3} \quad \Rightarrow \quad 1 = \frac{3}{3} = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots + (2k-1)^2 = \frac{k(4k^2-1)}{3} \quad \ldots\ldots\ldots \text{ (i)}$$

The statement for $n = k+1$ becomes

$$S(k+1): 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots + (2(k+1)-1)^2 = \frac{(k+1)(4(k+1)^2-1)}{3}$$

$$\Rightarrow \quad 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots + (2k+1)^2 = \frac{(k+1)(4(k^2+2k+1)-1)}{3}$$

$$= \frac{(k+1)(4k^2+8k+4-1)}{3}$$

$$= \frac{(k+1)(4k^2+8k+3)}{3}$$

$$= \frac{4k^3+8k^2+3k+4k^2+8k+3}{3}$$

$$= \frac{4k^3+12k^2+11k+3}{3}$$

Adding $(2k+1)^2$ on both sides of equation (i)

$$1^2 + 3^2 + 5^2 + \ldots\ldots\ldots + (2k-1)^2 + \left(2k+1\right)^2 = \frac{k(4k^2-1)}{3} + \left(2k+1\right)^2$$

$$\Rightarrow \quad 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots + \left(2k+1\right)^2 = \frac{k(4k^2-1)+3(2k+1)^2}{3}$$

$$= \frac{k(4k^2-1)+3(4k^2+4k+1)}{3}$$

$$= \frac{4k^3-k+12k^2+12k+3}{3}$$

$$= \frac{4k^3 + 12k^2 + 11k + 3}{3}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 20

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots\ldots\ldots + \binom{n+2}{3} = \binom{n+3}{4}$$

Solution

Suppose $\qquad S(n): \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots\ldots\ldots + \binom{n+2}{3} = \binom{n+3}{4}$

Put $n = 1$

$\text{L.H.S} = \binom{3}{3} = 1$

$\text{R.H.S} = \binom{1+3}{4} = \binom{4}{4} = 1$

$\text{L.H.S} = \text{R.H.S}$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots\ldots\ldots + \binom{k+2}{3} = \binom{k+3}{4} \qquad \ldots\ldots\ldots \text{(i)}$$

The statement for $n = k+1$ becomes

$$S(k+1): \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots\ldots\ldots + \binom{k+1+2}{3} = \binom{k+1+3}{4}$$

$$\Rightarrow \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots\ldots\ldots + \binom{k+3}{3} = \binom{k+4}{4}$$

Adding $\binom{k+3}{3}$ on both sides of equation (i)

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots\ldots\ldots + \binom{k+2}{3} + \binom{k+3}{3} = \binom{k+3}{4} + \binom{k+3}{3}$$

$$\Rightarrow \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots\ldots\ldots + \binom{k+3}{3} = \binom{k+3+1}{4} \qquad \because \binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

$$\Rightarrow \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots\ldots\ldots + \binom{k+3}{3} = \binom{k+4}{4}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 21

Prove by mathematical induction that for all positive integral values of n.

(i) $n^2 + n$ is divisible by 2   (ii) $5^n - 2^n$ is divisible by 3

(iii) $5^n - 1$ is divisible by 4   (iv) $8 \times 10^n - 2$ is divisible by 6

(v) $n^3 - n$ is divisible by 6

Solution

(i)   Suppose $S(n): n^2 + n$

Put $n = 1$

$S(1): 1^2 + 1 = 2$

$S(1)$ is clearly divisible by 2, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$S(k): k^2 + k$

Then there exists quotient Q such that

$k^2 + k = 2Q$

$$
\begin{array}{r}
Q \\
2\overline{)k^2 + k} \\
\underline{k^2 + k} \\
0
\end{array}
$$

The statement for $n = k + 1$

$S(k+1): (k+1)^2 + k + 1$

$\qquad = k^2 + 2k + 1 + k + 1$

$\qquad = k^2 + k + 2k + 2$

$\qquad = 2Q + 2k + 2 \qquad\qquad\qquad \because k^2 + k = 2Q$

$\qquad = 2(Q + k + 1)$

Clearly $S(k+1)$ is divisible by 2.

Since the truth for $n = k$ implies the truth for $n = k + 1$ therefore the given statement is true for $\forall\ n \in \mathbb{Z}^+$.

(ii)   Suppose $S(n): 5^n - 2^n$

Put $n = 1$

$S(1): 5^1 - 2^1 = 3$

$S(1)$ is clearly divisible by 3, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$S(k): 5^k - 2^k$

Then there exists quotient Q such that

$5^k - 2^k = 3Q$

The statement for $n = k + 1$

$S(k+1): 5^{k+1} - 2^{k+1}$

$\qquad = 5 \cdot 5^k - 2 \cdot 2^k$

$\qquad = 5 \cdot 5^k - 5 \cdot 2^k + 5 \cdot 2^k - 2 \cdot 2^k$

$\qquad = 5(5^k - 2^k) + 2^k(5 - 2)$

$\qquad = 5(3Q) + 2^k \cdot 3 \qquad\qquad \because 5^k - 2^k = 3Q$

$\qquad = 3(5Q + 2^k)$

Clearly $S(k+1)$ is divisible by 3.

Since the truth for $n = k$ implies the truth for $n = k + 1$ therefore the given statement

is true for $\forall\ n\in\mathbb{Z}^+$.

(iii)   *Same as Question # 21 (ii)*

$$\text{Hint:}\quad S(k+1):\ 5^{k+1}-1$$
$$= 5\cdot 5^k -1\quad = 5\cdot 5^k -5+5-1$$
$$= 5\left(5^k -1\right)+4\quad = 5(4Q)-4 \qquad \because\ 5^k -1=4Q$$

(iv)   Suppose $S(n):\ 8\times 10^n -2$

Put $n=1$

$$S(1):\ 8\times 10^1 -2=80-2=78=6\times 13$$

$S(1)$ is clearly divisible by 6, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$$S(k):\ 8\times 10^k -2$$

Then there exists quotient Q such that

$$8\times 10^k -2 = 6Q$$

The statement for $n=k+1$

$$S(k+1):\ 8\times 10^{k+1} -2$$
$$=8\times 10\cdot 10^k -2$$
$$= 8\times 10\cdot 10^k -2\cdot 10+2\cdot 10-2 \qquad -\text{ing \& +ing } 2\cdot 10$$
$$= 10\left(8\times 10^k -2\right)+20-2$$
$$= 10(6Q)+18 \qquad\qquad \because\ 8\times 10^k -2=6Q$$
$$= 6(10Q+3)$$

Clearly $S(k+1)$ is divisible by 6.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^+$.

(v)   Suppose $S(n):\ n^3 -n$

Put $n=1$

$$S(1):\ 1^3 -1=0$$

$S(1)$ i.e. 0 is clearly divisible by 6, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$$S(k):\ k^3 -k$$

Then there exists quotient Q such that

$$k^3 -k=6Q$$

The statement for $n=k+1$

$$S(k+1):\ (k+1)^3 -(k+1)$$
$$=k^3 +3k^2 +3k +1-k-1$$
$$=k^3 +3k^2 +3k -k$$
$$=(k^3 -k)+3(k^2 +k)$$
$$=6Q+3(2Q')$$
$$=6Q+6Q'$$

Since $n^2 +n$ is divisible by 2

Therefore $n^2 +n=2Q'$

Or $k^2 +k=2Q'$

Clearly $S(k+1)$ is divisible by 6.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n \in \mathbb{Z}^{+}$.

Question # 22

$$\frac{1}{3} + \frac{1}{3^2} + \ldots\ldots\ldots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$$

Solution   Suppose $S(n): \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots\ldots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$

Put $n = 1$

$$S(1): \frac{1}{3} = \frac{1}{2}\left(1 - \frac{1}{3^1}\right) \ \Rightarrow\ \frac{1}{3} = \frac{1}{2}\left(\frac{2}{3}\right) \ \Rightarrow\ \frac{1}{3} = \frac{1}{3}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots\ldots + \frac{1}{3^k} = \frac{1}{2}\left(1 - \frac{1}{3^k}\right) \ \ldots\ldots\ldots\ldots \text{ (i)}$$

The statement for $n = k+1$ becomes

$$S(k+1): \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots\ldots + \frac{1}{3^{k+1}} = \frac{1}{2}\left(1 - \frac{1}{3^{k+1}}\right)$$

Adding $\dfrac{1}{3^k}$ on both sides of equation (i)

$$\frac{1}{3} + \frac{1}{3^2} + \ldots\ldots\ldots + \frac{1}{3^k} + \frac{1}{3^{k+1}} = \frac{1}{2}\left(1 - \frac{1}{3^k}\right) + \frac{1}{3^{k+1}}$$

$$\Rightarrow\ \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots\ldots + \frac{1}{3^{k+1}} = \frac{1}{2} - \frac{1}{2 \cdot 3^k} + \frac{1}{3 \cdot 3^k}$$

$$= \frac{1}{2} - \frac{1}{3^k}\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{2} - \frac{1}{3^k}\left(\frac{3-2}{6}\right) \ = \frac{1}{2} - \frac{1}{3^k}\left(\frac{1}{6}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{3^k}\left(\frac{1}{3}\right)\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{3^{k+1}}\right)$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 23

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots\ldots + (-1)^{n-1} \cdot n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}$$

Solution   Suppose

$$S(n):\ 1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots\ldots + (-1)^{n-1} \cdot n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}$$

Put $n = 1$

$$S(1):\ 1^2 = \frac{(-1)^{1-1} \cdot 1(1+1)}{2} \quad \Rightarrow \quad 1 = \frac{(-1)^0 \cdot 2}{2} \quad \Rightarrow \quad 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k):\ 1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots\ldots + (-1)^{k-1} \cdot k^2 = \frac{(-1)^{k-1} \cdot k(k+1)}{2} \quad \ldots\ldots\ldots\ (i)$$

The statement for $n = k+1$ becomes

$$S(k+1):\ 1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots\ldots + (-1)^{k+1-1} \cdot (k+1)^2 = \frac{(-1)^{k+1-1} \cdot (k+1)(k+1+1)}{2}$$

$$\Rightarrow\ 1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots\ldots + (-1)^{k} \cdot (k+1)^2 = \frac{(-1)^{k} \cdot (k+1)(k+2)}{2}$$

Adding $(-1)^k \cdot (k+1)^2$ on both sides of equation (i)

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots\ldots + (-1)^{k-1} \cdot k^2 + (-1)^{k} \cdot (k+1)^2 = \frac{(-1)^{k-1} \cdot k(k+1)}{2} + (-1)^{k} \cdot (k+1)^2$$

$$\Rightarrow\ 1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots\ldots + (-1)^{k} \cdot (k+1)^2 = \frac{(-1)^{k-1} \cdot k(k+1) + 2(-1)^{k} \cdot (k+1)^2}{2}$$

$$= \frac{(-1)^{k}(k+1)\left[(-1)^{-1}k + 2(k+1)\right]}{2}$$

$$= \frac{(-1)^{k}(k+1)\left[-k + 2k + 2\right]}{2}$$

$$= \frac{(-1)^{k}(k+1)(k+2)}{2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 24

$$1^3 + 3^3 + 5^3 + \ldots\ldots\ldots + (2n-1)^3 = n^2(2n^2 - 1)$$

Solution   Suppose $S(n):\ 1^3 + 3^3 + 5^3 + \ldots\ldots\ldots + (2n-1)^3 = n^2(2n^2 - 1)$

Put $n = 1$

$$S(1):\ 1^3 = 1^2\left(2(1)^2 - 1\right) \quad \Rightarrow \quad 1 = 1(2 - 1) \quad \Rightarrow 1 = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k):\ 1^3 + 3^3 + 5^3 + \ldots\ldots\ldots + (2k-1)^3 = k^2(2k^2 - 1) \quad \ldots\ldots\ldots\ (i)$$

The statement for $n = k+1$ becomes

$S(k+1)$: $1^3 + 3^3 + 5^3 + \ldots\ldots\ldots\ldots + (2(k+1)-1)^3 = (k+1)^2\,(2(k+1)^2 - 1)$

$$\Rightarrow\ 1^3 + 3^3 + 5^3 + \ldots\ldots\ldots\ldots + (2k+1)^3 = (k^2 + 2k + 1)(2(k^2 + 2k + 1) - 1)$$

$$= (k^2 + 2k + 1)(2k^2 + 4k + 2 - 1)$$

$$= (k^2 + 2k + 1)(2k^2 + 4k + 1)$$

$$= 2k^4 + 4k^3 + 2k^2 + 4k^3 + 8k^2 + 4k + k^2 + 2k + 1$$

$$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

Adding $(2k+1)^3$ on both sides of equation (i)

$S(k)$: $1^3 + 3^3 + 5^3 + \ldots\ldots\ldots\ldots + (2k-1)^3 + (2k+1)^3 = k^2\,(2k^2 - 1) + (2k+1)^3$

$$\Rightarrow\ 1^3 + 3^3 + 5^3 + \ldots\ldots\ldots\ldots + (2k+1)^3 = k^2\,(2k^2 - 1) + (2k)^3 + 3(2k)^2(1) + 3(2k)(1)^2 + (1)^3$$

$$\Rightarrow\ 1^3 + 3^3 + 5^3 + \ldots\ldots\ldots\ldots + (2k+1)^3 = 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1$$

$$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true
for all positive integer n.

Question # 25

$x + 1$ is a factor of $x^{2n} - 1; (x \ne -1)$

Solution   Suppose $S(n)$: $x^{2n} - 1$

Put $n = 1$

$S(1)$: $x^{2(1)} - 1 = x^2 - 1 = (x-1)(x+1)$

$x + 1$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$S(k)$: $x^{2k} - 1$

Then there exists quotient Q such that

$$x^{2k} - 1 = (x+1)Q$$

The statement for $n = k + 1$

$S(k+1)$: $x^{2(k+1)} - 1$

$$= x^{2k+2} - 1$$

$$= x^{2k+2} - x^{2k} + x^{2k} - 1 \qquad\qquad \text{+ing and –ing } x^{2k}$$

$$= x^{2k}(x^2 - 1) + (x^{2k} - 1)$$

$$= x^{2k}(x-1)(x+1) + (x+1)Q \qquad\qquad \because\ x^{2k} - 1 = (x+1)Q$$

$$= (x+1)\left(x^{2k}(x-1) + Q\right)$$

Clearly $x + 1$ is a factor of $S(k+1)$.

Since the truth for $n = k$ implies the truth for $n = k + 1$ therefore the given statement
is true for $\forall\ n \in \mathbb{Z}^+$.

Question # 26

$x - y$ is a factor of $x^n - y^n; (x \ne y)$

Solution   Suppose $S(n)$: $x^n - y^n$

Put $n = 1$

$$S(1): x^1 - y^1 = x - y$$

$x - y$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$$S(k): x^k - y^k$$

Then there exists quotient Q such that

$$x^k - y^k = (x - y)Q$$

The statement for $n = k + 1$

$$S(k+1): x^{k+1} - y^{k+1}$$

$$= x \cdot x^k - y \cdot y^k$$

$$= x \cdot x^k - x \cdot y^k + x \cdot y^k - y \cdot y^k \qquad \text{–ing \& +ing } x\,y^k$$

$$= x(x^k - y^k) + y^k(x - y)$$

$$= x(x - y)Q + y^k(x - y) \qquad \because x^k - y^k = (x - y)Q$$

Clearly $x - y$ is a factor of $S(k+1)$.

Since the truth for $n = k$ implies the truth for $n = k + 1$ therefore the given statement is true for $\forall \ n \in \mathbb{Z}^+$.

Question # 27

$x + y$ is a factor of $x^{2n-1} + y^{2n-1}; (x \neq y)$

Solution   Suppose $S(n): x^{2n-1} + y^{2n-1}$

Put $n = 1$

$$S(1): x^{2(1)-1} + y^{2(1)-1} = x^1 + y^1 = x + y$$

$x + y$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$$S(k): x^{2k-1} + y^{2k-1}$$

Then there exists quotient Q such that

$$x^{2k-1} + y^{2k-1} = (x + y)Q$$

The statement for $n = k + 1$

$$S(k+1): x^{2(k+1)-1} + y^{2(k+1)-1}$$

$$= x^{2k+2-1} + y^{2k+2-1}$$

$$= x^{2k+2-1} - x^{2k-1}y^2 + x^{2k-1}y^2 + y^{2k+2-1} \qquad \text{+ing and –ing } x^{2k-1}y^2$$

$$= x^{2k-1}(x^2 - y^2) + y^2(x^{2k-1} + y^{2k-1})$$

$$= x^{2k-1}(x - y)(x + y) + y^2(x + y)Q \qquad \because x^{2k-1} + y^{2k-1} = (x + y)Q$$

$$= (x + y)\left(x^{2k-1}(x - y) + y^2 Q\right)$$

Clearly $x + y$ is a factor of $S(k+1)$.

Since the truth for $n = k$ implies the truth for $n = k + 1$ therefore the given statement is true for $\forall \ n \in \mathbb{Z}^+$.

Principle of Extended Mathematical Induction

A given statement $S(n)$ is true for $n \geq i$ if the following two conditions hold

Condition I: $S(i)$ is true i.e. $S(n)$ is true for $n = i$ and

Condition II: $S(k+1)$ is true whenever $S(k)$ is true for any positive integer k,

Then $S(n)$ is true for all positive integers

Question # 28

Use mathematical induction to show that

$1 + 2 + 2^2 + \ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 1$ for all non-negative integers n.

Solution   Suppose $S(n)\colon 1 + 2 + 2^2 + \ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 1$

Put $n = 0$

$S(1)\colon 1 = 2^{0+1} - 1 = \Rightarrow 1 = 2 - 1 \Rightarrow 1 = 1$

$\boxed{\textit{Note}: \text{Non- negative number are } 0, 1, 2, 3, \ldots\ldots\ldots}$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$S(k)\colon 1 + 2 + 2^2 + \ldots\ldots\ldots\ldots + 2^k = 2^{k+1} - 1$ …………. (i)

The statement for $n = k+1$ becomes

$S(k+1)\colon 1 + 2 + 2^2 + \ldots\ldots\ldots\ldots + 2^{k+1} = 2^{k+1+1} - 1$

$$= 2^{k+2} - 1$$

Adding 2^{k+1} on both sides of equation (i)

$1 + 2 + 2^2 + \ldots\ldots\ldots\ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$

$\Rightarrow 1 + 2 + 4 + \ldots\ldots\ldots\ldots + 2^{k+1} = 2(2^{k+1}) - 1 \qquad \because 2^{k+1} + 2^{k+1} = 2(2^{k+1})$

$$= 2^{k+1+1} - 1$$
$$= 2^{k+1+1} - 1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all non-negative integers n.

Question # 29

If A and B are square matrices and $AB = BA$, then show by mathematical induction that $AB^n = B^n A$ for any positive integer n.

Solution   Suppose $S(n)\colon AB^n = B^n A$

Put $n = 1$

$S(1)\colon AB^1 = B^1 A \Rightarrow AB = BA$

$S(1)$ is true as we have given $AB = BA$, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$S(k)\colon AB^k = B^k A$ ……………. (i)

The statement for $n = k+1$

$S(k+1)\colon AB^{k+1} = B^{k+1} A$

Post-multiplying equation (i) by B.

$$(AB^k)B = (B^k A)B$$

$\Rightarrow A(B^k B) = B^k (AB)$ by associative law

$\Rightarrow AB^{k+1} = B^k (BA)$ $\because AB = BA$ (*given*)

$$= (B^k B)A = B^{k+1} A$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integers n.

Question # 30

Prove by Principle of mathematical induction that $n^2 - 1$ is divisible by 8 when n is and odd positive integer.

Solution   Suppose $S(n)$: $n^2 - 1$

  Put $n = 1$

       $S(1)$: $(1)^2 - 1 = 0$

 $S(1)$ is clearly divisible by 8, Thus condition I is satisfied

  Now suppose that given statement is true for $n = k$ where k is odd.

         $S(k)$: $k^2 - 1$

  Then there exists quotient Q such that

           $k^2 - 1 = 8\,Q$

  As $k + 2$ is the next odd integer after k The statement for $n = k + 1$

$$S(k+2): \left(k+2\right)^2 - 1$$
$$= k^2 + 4k + 4 - 1$$
$$= k^2 - 1 + 4k + 4$$
$$= 8Q + 4(k+1) \qquad\qquad \because \; k^2 + k = 2Q$$

  Since k is odd therefore $k + 1$ is even so their exists integer t such that $k + 1 = 2t$

$$\Rightarrow \; S(k+2) := 8Q + 4(2t)$$
$$= 8Q + 8t$$

  Clearly $S(k + 2)$ is divisible by 8 so condition II is satisfied.

  Therefor the given statement is true for odd positive integers.

Question # 31

Use the principle of mathematical induction to prove that $\ln x^n = n \ln x$ for any integral $n \geq 0$ if x is a positive number.

Solution   Suppose $S(n)$: $\ln x^n = n \ln x$

  Put $n = 1$

       $S(1)$: $\ln x^1 = (1) \ln x$ $\Rightarrow \; \ln x = \ln x$

 $S(1)$ is true so condition I is satisfied.

  Now suppose that given statement is true for $n = k$

       $S(k)$: $\ln x^k = k \ln x$ …………… (i)

  The statement for $n = k + 1$

       $S(k + 1)$: $\ln x^{k+1} = (k+1) \ln x$

  Now adding $\ln x$ on both sides of equation (i)

$$\ln x^k + \ln x = k \ln x + \ln x$$
$$\Rightarrow \; \ln x^k \cdot x = (k+1) \ln x \qquad\qquad \because \; \ln x + \ln y = \ln x\,y$$
$$\Rightarrow \; \ln x^{k+1} = (k+1) \ln x$$

  Thus $S(k + 1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true

  for all $n \in \mathbb{Z}^+$.

Question # 32

$n! > 2^n - 1$ for integral values of $n \geq 4$.

Solution   Suppose $S(n): n! > 2^n - 1$; $n \geq 4$

Put $n = 4$

$S(4): 4! > 2^4 - 1 \Rightarrow 24 > 16 - 1 \Rightarrow 24 > 15$

$S(4)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$S(k): k! > 2^k - 1 \ldots\ldots\ldots\ldots$ (i)

The statement for $n = k + 1$

$S(k+1): (k+1)! > 2^{k+1} - 1$

Multiplying both sides of equation (i) by $k + 1$

$(k+1)k! > (k+1)(2^k - 1)$

$\Rightarrow (k+1)! > (k+1+2-2)(2^k - 1)$ $\because (k+1)k! = (k+1)!$

$\Rightarrow (k+1)! > (k-1+2)(2^k - 1)$

$\Rightarrow (k+1)! > k \cdot 2^k - k - 2^k + 1 + 2 \cdot 2^k - 2$

$\Rightarrow (k+1)! > (k \cdot 2^k - 2^k - k) + 2^{k+1} - 1$

$\Rightarrow (k+1)! > 2^{k+1} - 1$ $\because k \cdot 2^k - 2^k - k \geq 0 \ \ \forall \ k \geq 4$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 4$.

Question # 33

$n^2 > n + 3$ for integral values of $n \geq 3$.

Solution   Suppose $S(n): n^2 > n + 3$; $n \geq 3$

Put $n = 3$

$S(3): 3^2 > 3 + 3 \Rightarrow 9 > 6$

$S(3)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$S(k): k^2 > k + 3 \ldots\ldots\ldots\ldots$ (i)

The statement for $n = k + 1$

$S(k+1): (k+1)^2 > k + 1 + 3 \Rightarrow (k+1)^2 > k + 4$

Adding $2k + 1$ on both sides of equation (i)

$k^2 + 2k + 1 > k + 3 + 2k + 1$

$\Rightarrow (k+1)^2 > k + 4 + 2k$

$\Rightarrow (k+1)^2 > k + 4$ ignoring $2k$ as $2k > 0$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 3$.

Question # 34

$4^n > 3^n + 2^{n-1}$ for integral values of $n \geq 2$.

Solution   Suppose $S(n): 4^n > 3^n + 2^{n-1}$; $n \geq 2$

Put $n = 2$

$S(2): 4^2 > 3^2 + 2^{2-1} \Rightarrow 16 > 9 + 2 \Rightarrow 16 > 11$

$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

   $S(k):\ 4^k > 3^k + 2^{k-1}$ ……………. (i)

The statement for $n = k+1$

   $S(k+1):\ 4^{k+1} > 3^{k+1} + 2^{k+1-1}$

   $\Rightarrow\ 4^{k+1} > 3^{k+1} + 2^k$

Multiplying both sides of equation (i) by 4.

$$4(4^k) > 4(3^k + 2^{k-1})$$

$$\Rightarrow\ 4^{k+1} > 4 \cdot 3^k + 4 \cdot 2^{k-1}$$

$$\Rightarrow\ 4^{k+1} > (3+1) \cdot 3^k + (2+2) \cdot 2^{k-1}$$

$$\Rightarrow\ 4^{k+1} > 3 \cdot 3^k + 3^k + 2 \cdot 2^{k-1} + 2 \cdot 2^{k-1}$$

$$\Rightarrow\ 4^{k+1} > 3^{k+1} + 2^k + (3^k + 2^k)$$

$$\Rightarrow\ 4^{k+1} > 3^{k+1} + 2^k \qquad \text{ignoring } 3^k + 2^k \text{ as } 3^k + 2^k > 0$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 3$.

Question # 35

   $3^n < n!$ for integral values of $n \geq 6$.

Solution Suppose $S(n):\ 3^n < n!$; $n > 6$

Put $n = 7$

   $S(7):\ 3^7 < 7!\ \ \Rightarrow\ 2187 < 5040$

$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

   $S(k):\ 3^k < k!$……………. (i)

The statement for $n = k+1$

   $S(k+1):\ 3^{k+1} < (k+1)!$

Multiplying both sides of equation (i) by $k+1$.

$$(k+1)3^k < (k+1)k!$$

$$\Rightarrow\ \big((k-2)+3\big)3^k < (k+1)!$$

$$\Rightarrow\ (k-2)3^k + 3^{k+1} < (k+1)!$$

$$\Rightarrow\ 3^{k+1} < (k+1)! \qquad \because\ (k-2)3^k > 0\ \ \forall\ k > 6$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n > 6$.

Question # 36

   $n! > n^2$ for integral values of $n \geq 4$.

Solution Suppose $S(n):\ n! > n^2$; $n \geq 4$

Put $n = 4$

   $S(4):\ 4! > 4^2\ \ \Rightarrow\ 24 > 16$

$S(4)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$$S(k): \quad k! > k^2 \ldots\ldots\ldots\ldots \text{ (i)}$$

The statement for $n = k + 1$

$$S(k+1): \quad (k+1)! > (k+1)^2$$

Multiplying both sides of equation (i) by $k + 1$.

$$(k+1)k! > (k+1)k^2$$

$$\Rightarrow \quad (k+1)! > (k+1)(k+1) \qquad \because \quad k+1 < k^2 \quad \forall \quad k \geq 4$$

$$\Rightarrow \quad (k+1)! > (k+1)^2$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 4$.

Question # 37

$$3 + 5 + 7 + \ldots\ldots\ldots\ldots + (2n+5) = (n+2)(n+4) \text{ for integral values of } n \geq -1.$$

Solution   Suppose $S(n): 3 + 5 + 7 + \ldots\ldots\ldots\ldots + (2n+5) = (n+2)(n+4)$; $n \geq -1$

Put $n = -1$

$$S(-1): \quad 3 = (-1+2)(-1+4) \quad \Rightarrow \quad 3 = (1)(3) \quad \Rightarrow \quad 3 = 3$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k): \quad 3 + 5 + 7 + \ldots\ldots\ldots\ldots + (2k+5) = (k+2)(k+4) \quad \ldots\ldots\ldots\ldots \text{ (i)}$$

The statement for $n = k + 1$ becomes

$$S(k+1): \quad 3 + 5 + 7 + \ldots\ldots\ldots\ldots + (2(k+1)+5) = \big((k+1)+2\big)\big((k+1)+4\big)$$

$$\Rightarrow \quad 3 + 5 + 7 + \ldots\ldots\ldots\ldots + (2k+7) = (k+3)(k+5)$$

Adding $(2k+7)$ on both sides of equation (i)

$$S(k): \quad 3 + 5 + 7 + \ldots\ldots\ldots\ldots + (2k+5) + (2k+7) = (k+2)(k+4) + (2k+7)$$

$$\Rightarrow \quad 3 + 5 + 7 + \ldots\ldots\ldots\ldots + (2k+7) = k^2 + 2k + 4k + 8 + 2k + 7$$

$$= k^2 + 8k + 15$$

$$= k^2 + 5k + 3k + 15$$

$$= k(k+5) + 3(k+5)$$

$$= (k+5)(k+3)$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq -1$.

Question # 38

$$1 + nx \leq (1+x)^n \text{ for integral values of } n \geq -1.$$

Solution   Suppose $S(n): 1 + nx \leq (1+x)^n$; $n \geq 2$

Put $n = 2$

$$S(2): \quad 1 + 2x \leq (1+x)^2 \quad \Rightarrow \quad 1 + 2x \leq 1 + 2x + x^2$$

$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$$S(k): \quad 1 + kx \leq (1+x)^k \ldots\ldots\ldots\ldots \text{ (i)}$$

The statement for $n = k + 1$

$$S(k+1): \quad 1 + (k+1)x \leq (1+x)^{k+1}$$

Multiplying both sides of equation (i) by $1 + x$.

$$(1+kx)(1+x) \le (1+x)^k(1+x)$$

$$\Rightarrow \quad 1+kx+x+kx^2 \le (1+x)^{k+1}$$

$$\Rightarrow \quad 1+kx+x \le (1+x)^{k+1} \qquad\qquad \because \ kx^2 > 0$$

$$\Rightarrow \quad 1+(k+1)x \le (1+x)^{k+1}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \ge 2$

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